

Problem 1: Columbus, Ohio, sits at around 40° north. If we draw a circle around the earth at this latitude, is it true that there must be two points on this circle, diametrically opposed to each other, which are the same temperature at the same time?

Solution: Fix a particular time, and let $T(\theta)$ denote the temperature at the point on the circle of latitude whose longitude is θ , where $0 \leq \theta < 2\pi$. We assume that temperature varies continuously as we move around the circle.

Points that are diametrically opposite on this circle differ in longitude by π . Define

$$f(\theta) = T(\theta) - T(\theta + \pi).$$

The function f is continuous. Moreover,

$$f(\theta + \pi) = T(\theta + \pi) - T(\theta + 2\pi) = T(\theta + \pi) - T(\theta) = -f(\theta),$$

since $T(\theta + 2\pi) = T(\theta)$.

Now choose any longitude θ_0 . If $f(\theta_0) = 0$, then

$$T(\theta_0) = T(\theta_0 + \pi),$$

so the two diametrically opposite points have the same temperature.

Otherwise, $f(\theta_0)$ and $f(\theta_0 + \pi) = -f(\theta_0)$ have opposite signs. Since f is continuous, the Intermediate Value Theorem implies that there is some

$$\theta^* \in (\theta_0, \theta_0 + \pi)$$

such that

$$f(\theta^*) = 0.$$

Thus

$$T(\theta^*) = T(\theta^* + \pi).$$

Therefore, at any fixed time, there must be two diametrically opposite points on the circle of latitude at 40° north that have the same temperature.

Problem 2: I choose two nonnegative integers N and M (at random), and write them on two separate sheets of paper. The distributions of these choices are unknown to you, but you do know that the numbers are different. You choose one of the sheets at random and observe the number on it. The other number is still unknown to you. Your task is to guess whether the number you observed is larger or smaller than the number still unobserved. You have access to a random number generator (ie you can generate numbers on $[0, 1]$ uniformly at will). Devise a strategy which guesses which number is larger with greater than $1/2$ probability.

Solution: Let the two distinct nonnegative integers be N and M . We do not know their distributions.

Before looking at either sheet, generate a random integer K independently of N and M such that every nonnegative integer has positive probability of being chosen. For example, let

$$\mathbb{P}(K = k) = \frac{1}{2^{k+1}}, \quad k = 0, 1, 2, \dots$$

Such a K can be generated from a uniform random variable U on $(0, 1)$ by taking

$$K = \lfloor -\log_2 U \rfloor.$$

Now choose one of the two sheets uniformly at random and suppose the observed number is X . Use the following strategy:

$$\begin{cases} \text{if } X \leq K, \text{ then guess that } X \text{ is the smaller number,} \\ \text{if } X > K, \text{ then guess that } X \text{ is the larger number.} \end{cases}$$

To analyze the strategy, condition on the two numbers. Without loss of generality, suppose

$$N < M.$$

If

$$N \leq K < M,$$

then the strategy is always correct. Indeed, if we observe N , then $N \leq K$, so we correctly guess that it is the smaller number; if we observe M , then $M > K$, so we correctly guess that it is the larger number.

If $K < N$ or $K \geq M$, then the strategy is correct for exactly one of the two sheets. Since the sheet is chosen uniformly at random, the conditional probability of success in this case is $1/2$.

Therefore,

$$\begin{aligned} \mathbb{P}(\text{correct} \mid N, M) &= 1 \cdot \mathbb{P}(N \leq K < M) + \frac{1}{2}(1 - \mathbb{P}(N \leq K < M)) \\ &= \frac{1}{2} + \frac{1}{2} \mathbb{P}(N \leq K < M). \end{aligned}$$

Because $N < M$ are distinct integers and every nonnegative integer has positive probability under the distribution of K ,

$$\mathbb{P}(N \leq K < M) > 0.$$

Hence

$$\mathbb{P}(\text{correct} \mid N, M) > \frac{1}{2}.$$

Thus, regardless of the unknown distributions of N and M , this strategy guesses correctly with probability strictly greater than $1/2$.

Problem 3: Choose at random 3 points on the unit circle $x^2 + y^2 = 1$. Interpret them as cuts that divide the circle into three arcs. Compute the expected length of the arc that contains the point $(1, 0)$.

Remark: Here is a “solution”. Let L_1, L_2, L_3 be the lengths of the three arcs. Then $L_1 + L_2 + L_3 = 2\pi$ and by symmetry $E(L_1) = E(L_2) = E(L_3)$, so the answer is $E(L_1) = 2\pi/3$. Explain why this is wrong.

Solution: Let the point $(1, 0)$ correspond to angle 0, and let the three randomly chosen points have angles

$$\Theta_1, \Theta_2, \Theta_3,$$

chosen independently and uniformly from $[0, 2\pi)$.

The arc containing $(1, 0)$ extends from the closest cut counterclockwise from $(1, 0)$ to the closest cut clockwise from $(1, 0)$.

Let X be the distance from $(1, 0)$ to the closest cut in the clockwise direction. Then

$$X = \min\{\Theta_1, \Theta_2, \Theta_3\}.$$

For $0 \leq x \leq 2\pi$,

$$\begin{aligned} \mathbb{P}(X > x) &= \mathbb{P}(\Theta_1 > x, \Theta_2 > x, \Theta_3 > x) \\ &= \left(\frac{2\pi - x}{2\pi}\right)^3. \end{aligned}$$

Therefore, using the tail integral formula,

$$\begin{aligned} \mathbb{E}[X] &= \int_0^{2\pi} \mathbb{P}(X > x) dx \\ &= \int_0^{2\pi} \left(1 - \frac{x}{2\pi}\right)^3 dx \\ &= \frac{2\pi}{4} = \frac{\pi}{2}. \end{aligned}$$

Similarly, let Y be the distance from $(1, 0)$ to the closest cut in the counterclockwise direction. By symmetry,

$$\mathbb{E}[Y] = \frac{\pi}{2}.$$

The length L of the arc containing $(1, 0)$ is

$$L = X + Y.$$

Hence, by linearity of expectation,

$$\boxed{\mathbb{E}[L] = \mathbb{E}[X] + \mathbb{E}[Y] = \frac{\pi}{2} + \frac{\pi}{2} = \pi.}$$

The proposed argument giving $2\pi/3$ fails because it computes the expected length of a *uniformly selected arc*, whereas the arc containing $(1, 0)$ is not uniformly selected from the three arcs.

Indeed, longer arcs are more likely to contain $(1, 0)$. Thus the arc containing a fixed point is *length-biased*: an arc of length ℓ has probability proportional to ℓ of being the one containing the fixed point. Consequently, although each of the three arcs has expected length $2\pi/3$, the arc containing $(1, 0)$ has the larger expected length π .

Remark. This problem failed to define random. In our solution, we interpret random to mean independently and uniformly at random from the unit circle (ie as if a uniform distribution were specified on the angle, Θ). If the points are not chosen independently, then the answer need not be π .

Likewise, if you do not assume a uniform distribution of an angle, then you may get a different result. For instance, one may interpret random as choosing a point $x \in [-1, 1]$ uniformly, and then taking $y = \pm\sqrt{1-x^2}$ with equal probability. Then the points are uniform in cartesian coordinates, but not uniform around the circle. For three independent points chosen this way, the expected length of the arc is $17\pi/16$! We'll leave this computation to the reader. For a similar seemingly paradoxical problem, see Bertrand's Paradox.

Problem 4: Intuitively the average (mean) of a random variable is somewhere in the middle of the possible values it takes. This is not true. Give an example of a random variable whose average is infinity, though the random variable takes finite values with probability 1.

Solution: Consider the following gambling game. We repeatedly toss a fair coin until the first head appears. If the first head occurs on the n th toss, then you win

$$X = 2^n$$

dollars.

For the first head to occur on toss n , the first $n - 1$ tosses must all be tails and the n th toss must be a head. Thus

$$\mathbb{P}(X = 2^n) = \left(\frac{1}{2}\right)^{n-1} \left(\frac{1}{2}\right) = \frac{1}{2^n}, \quad n = 1, 2, 3, \dots$$

The game ends after finitely many tosses with probability 1, since the probability of never obtaining a head is

$$\lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^n = 0.$$

Therefore X is finite with probability 1.

However, the expected winnings are

$$\begin{aligned} \mathbb{E}[X] &= \sum_{n=1}^{\infty} 2^n \mathbb{P}(X = 2^n) \\ &= \sum_{n=1}^{\infty} 2^n \frac{1}{2^n} \\ &= \sum_{n=1}^{\infty} 1 \\ &= \infty. \end{aligned}$$

Hence $\mathbb{E}[X] = \infty$ even though the amount won is finite with probability 1.

How much would you be willing to pay to play this game? After all, you will always win at least \$2. Prof. Holmes uses this example in Math 3589: Introduction to Financial Mathematics, to introduce utility functions and describe risk-averse investors. The class always has a couple students willing to give all of their money to play the game, even though the probability of winning more than \$200 is less than 1%!